Puzzle of the Week

Pan Balance With Weights – 3

A pan balance tells you when its two sides are carrying the same amount of weight or whether one side is heavier than the other.

THE CHALLENGE: You have a very large collection of 4-ounce and 7-ounce weights to use on both sides of a pan balance. You can weigh a 3-ounce item by using a 4-ounce weight together with the item on one side and a 7-ounce weight on the other. Which weights can you weigh exactly and which ones can't you weigh exactly?

EXPLORATION: How do your results change if you have 5- and 9-ounce weights? How about other pairs of weights that have no common divisor larger than 1? Can you find any patterns in your data? What would happen if you had three kinds of weights to work with - say 3 ounces, 4 ounces, and 7 ounces?

Puzzle of the Week

Pan Balance With Weights – 3 – Notes

THE CHALLENGE: Allowing weights on both sides of the pan balance is like allowing positive and negative multiples of both numbers. For example, suppose we weigh an item by putting the item with one 7-ounce weights on one side and three 4-ounce weights on the other side. The equation is <item> + 1 x 7 = 3 x 4. Putting the multiple of 7 on the other side gives <item> = 3 x 4 - 1 x 7 = 5. We have created a negative multiple of 7 by putting it on the same pan with the item.

Therefore, we are looking at all numbers that can be created by adding any multiple (positive, zero, or negative) of 4 to any multiple of 7. This is called Bezout's Theorem. The theorem states that the set of all possible sums of the multiples of two numbers is exactly the set of every multiple of their greatest common divisor. Because the greatest common divisor of 4 and 7 is 1, in this case we can weigh every multiple of 1 ounce, which is all numbers.

One way to see that this is true is to use the Chicken McNugget Theorem from "Pan Balance with Weights - 1." Suppose the two numbers are n and m. From that theorem, we know that using only nonnegative multiples, we can hit every weight starting with (n - 1) x (m - 1). In particular, we can write the weight (n x m) - 1 as a sum of a multiple of n and a multiple of n. Then 1 = [(n x m) - 1] - n x m tells us that 1 can be written as a sum of multiples of n and m.

EXPLORATION: Because 5 and 9 have a greatest common divisor of 1, we can weigh all amounts with these weights too.

If you had three weights that had at least one pair of numbers that had a greatest common divisor of 1, you would be able to weigh all possible weights and you would have more choices of how to do it.