

Puzzle of the Week

Product Equals Sum

THE CHALLENGE: Suppose you have five positive whole numbers, A, B, C, D, and E, so that the following is true. What is the largest possible value of any of these numbers?

$$A \times B \times C \times D \times E = A + B + C + D + E$$

EXPLORATION: What would happen if you had a different number of numbers in this kind of equation?

Puzzle of the Week

Product Equals Sum – Notes

THE CHALLENGE: As you stare at the problem, you realize you want to get some leverage over the fact that, as the numbers increase, the sum will not grow nearly as fast as the product. One simple thing to do is to turn the sum into a product as follows.

Assume that the numbers, A, B, C, D, and E, are in decreasing order so that A is the largest value (it may be tied with others). Then $A + B + C + D + E \leq 5A$. Consequently, $A \times B \times C \times D \times E = A + B + C + D + E \leq 5A$. Therefore, we have $B \times C \times D \times E \leq 5$. As these are whole numbers, that leaves us with very few possibilities to look at!

Each of these will be listed in B, C, D, E order.

- 5, 1, 1, 1: Then $A \times 5 \times 1 \times 1 \times 1 = A + 5 + 1 + 1 + 1$, so $5A = A + 8$. This would mean A is 2, which can't happen because A is supposed to be the largest value.
- 4, 1, 1, 1: Then $A \times 4 \times 1 \times 1 \times 1 = A + 4 + 1 + 1 + 1$, so $4A = A + 7$. This would mean $3A = 7$, which is impossible.
- 3, 1, 1, 1: Then $A \times 3 \times 1 \times 1 \times 1 = A + 3 + 1 + 1 + 1$, so $3A = A + 6$. This would mean $2A = 6$, so $A = 3$. This gives the solution $3 \times 3 \times 1 \times 1 \times 1 = 3 + 3 + 1 + 1 + 1 = 9$.
- 2, 1, 1, 1: Then $A \times 2 \times 1 \times 1 \times 1 = A + 2 + 1 + 1 + 1$, so $2A = A + 5$. This would mean $A = 5$. This gives the solution that $5 \times 2 \times 1 \times 1 \times 1 = 5 + 2 + 1 + 1 + 1 = 10$.
- 1, 1, 1, 1: Then $A \times 1 \times 1 \times 1 \times 1 = A + 1 + 1 + 1 + 1$, so $A = A + 4$, which is impossible.
- 2, 2, 1, 1: Then $A \times 2 \times 2 \times 1 \times 1 = A + 2 + 2 + 1 + 1$, so $4A = A + 6$. This would mean $A = 2$. This gives the solution that $2 \times 2 \times 2 \times 1 \times 1 = 2 + 2 + 2 + 1 + 1 = 8$.

Thus, there are three solutions: 3, 3, 1, 1, 1; 5, 2, 1, 1, 1; and 2, 2, 2, 1, 1. The largest number that appears in any of these is 5.

EXPLORATION: A very similar analysis can be done for any number of numbers. With larger numbers comes more to analyze.

For 2, 3, and 4 numbers we get:

- $2 \times 2 = 2 + 2$
- $3 \times 2 \times 1 = 3 + 2 + 1$
- $4 \times 2 \times 1 \times 1 = 4 + 2 + 1 + 1$

Beyond these specific answers, after doing a number of examples it is "clear" that when there are n numbers, the largest number will always be "n" associated with the solution $n \times 2 \times 1 \times 1 \times \dots \times 1 = n + 2 + 1 + 1 + \dots + 1$.